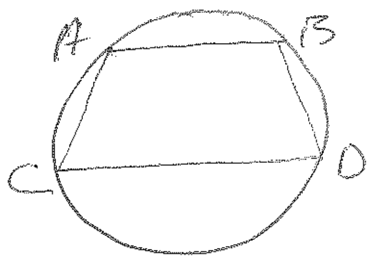
Maths Enrichment, UCD 6th February 2016

Geometry

We discussed the definitions, theorems and corollaries in section 6.11 (Circles) on pages 80 - 86 of the Leaving Certificate Mathematics Syllabus of An Roinn Oideachais agus Scileanna. This is available on line. The notation throughout is that used there.

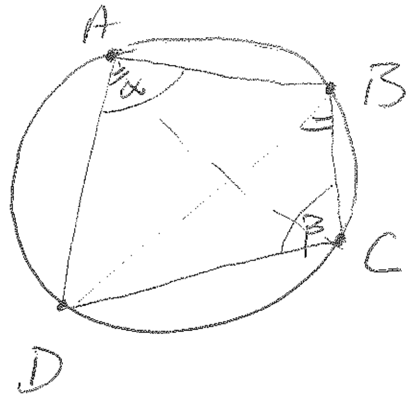
Based on that material, we showed that:

- two chords of equal length stand on arcs that subtend angles of equal measure at the circle.
- if two chords AB and CD of a circle are parallel then $|AC|=|BC|$ and a trapezium with the two non-parallel sides of equal length is a cyclic quadrilateral.



We also emphasised that this material contained the following:

- A quadrilateral is cyclic $\iff$ its opposite angles sum to 180°.
- A quadrilateral $ABCD$ is cyclic $\iff$ $|\angle DAC|=|\angle DBC|$.



$\alpha + \beta = 180^0$.

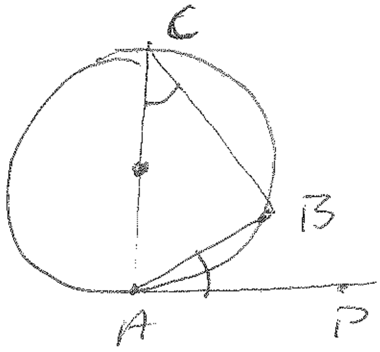
$|D\hat{A}C| = |D\hat{B}C|$.

Other theorems and problems

1. **The Alternate Segment Theorem**

   The angle between a tangent to a circle and a chord at the ponit of contact is equal to the angle in the alternate segment. (See diagram below.)

   In the proof, we make use of the fact that all angles in the alternate segment are equal (by Cor 2 , page 81, of Theorem 19) and show that $|\angle PAB| = |\angle ACB|$, where $|AC|$ is a diameter of the circle.

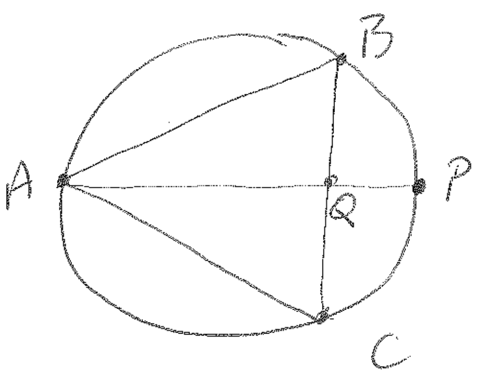


2. Let ABC be an equilateral triangle and let P be the point (other than A) where the bisector of $\angle BAC$ intersects the circumcircle of the triangle ABC. Then $|PA| = |PB| + |PC|$.

   Proof: Let Q denote the point where AP and BC intersect. Now, in the triangles ABQ and ACQ, $|AB| = |AC|$, $[AQ]$ is a common side and $|\angle BAQ| = |\angle CAQ| = 30°$, hence, the triangles ABQ and ACQ are congruent. So, $[AP]$ bisects $[BC]$ and it follows from Theorem 21, page 85, that $[AP]$ is a diameter. Therefore $\angle ABP$ is a right angle (by Cor 3 page 82). Let r denote the radius length. Now

$$\frac{1}{2} = \sin 30° = \sin(\angle BAQ) = \frac{|PB|}{|PA|} = \frac{|PB|}{2r} \Rightarrow |PB| = r.$$

   Similarly, $|PC| = r$ and the result follows.

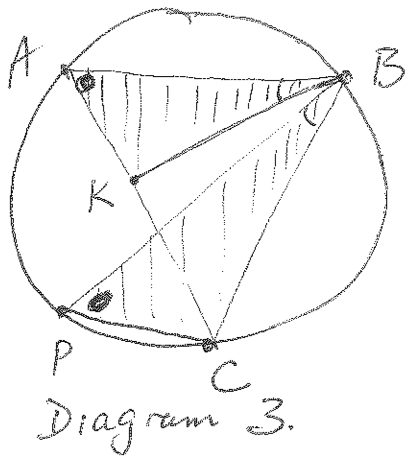


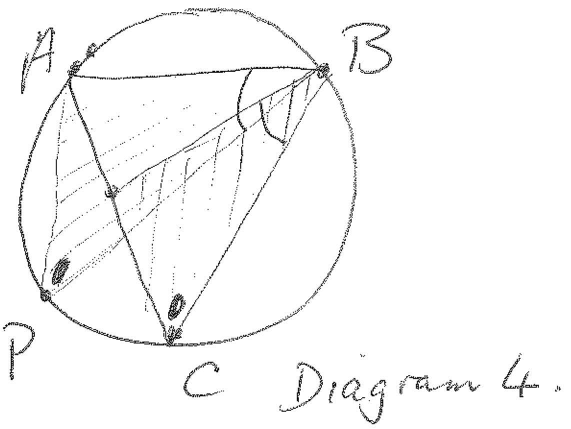
The last problem is just a special case of the next one.

3. Now suppose that ABC is an equilateral triangle and P is any point on its circumcircle. The distance from P to the furthest vertex of the the triangle is equal to the sum of the distances from P to the other two vertices.

   In the diagram, the furthest vertex from P is B, so we want to prove that

$$|PB| = |PA| + |PC|.$$





Proof: We choose a point K on the chord $[AC]$ (internally) so that $|\angle ABK| = |\angle PBC|$. Now consider the trianges ABK and PBC in Diagram 3 above. By Cor 2 of Theorem 19 (page 81), $|\angle BAK| = |\angle BPC|$ since these angles stand on the same arc BC, and by our choice of K, $|\angle ABK| = |\angle PBC|$, so these two triagles are similar (equiangular). Therefore, their sides are in proportion and

$$\frac{|AB|}{|PB|} = \frac{|AK|}{|PC|} \Rightarrow |AB||PC| = |PB||AK|. \qquad (1)$$

Similarly, if we consider the trianges ABP and KBC in Diagram 4 , we see that $|\angle BPA| = |\angle BCA|$ since they both stand on the arc AB, and

$|\angle ABK| + |\angle PBK| = |\angle PBC| + |\angle PBK|$, that is, $|\angle ABP| = |\angle KBC|$. Thus, these two triangels are similar and so

$$\frac{|PB|}{|BC|} = \frac{|AP|}{|KC|} \Rightarrow |BC||AP| = |PB||KC|. \qquad (2)$$

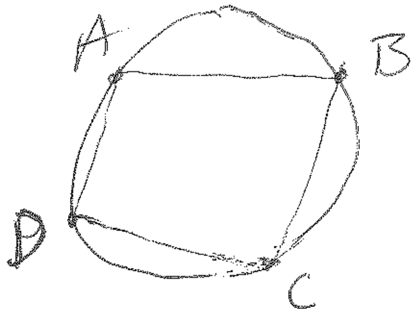
We'll denote the side length by l; that is, $l = |AB| = |BC| = |AC|$. Adding equations (1) and (2),

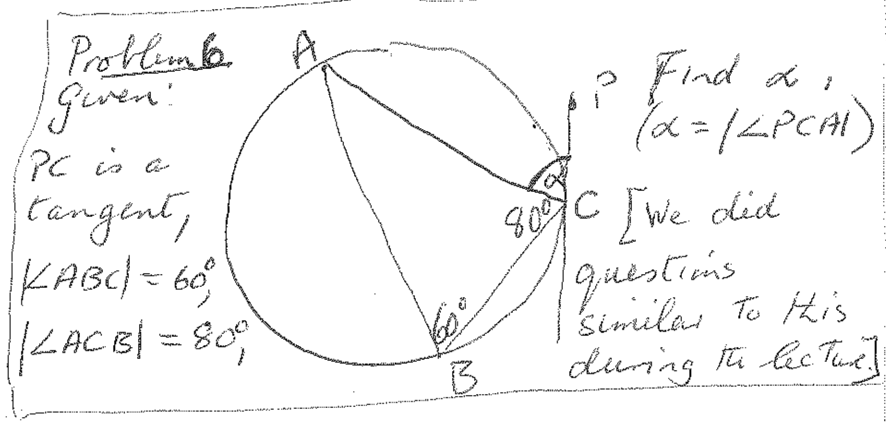
$$l(|PC|) + l(|AP|) = |PB|(|AK| + |KC|) = |PB||AC| = l(|PB|).$$

Thus, $|PB| = |PA| + |PC|$ as required.

The lecture ended here, just before we found another (necessary and sufficient) condition for a quadrilateral to be cyclic, which I promised at the beginning of the lecture. Ptolemy's theorem and its converse give that condition. The Greek mathematician and astronomer Claudius Ptolemy (100-170 AD) lived in Alexandria.

4. Ptolemy's theorem: If a quadrilateral is cyclic then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides. In the diagram below where A, B, C and D are the vertices of a cyclic quadrilateral, this means that $|AC||BD| = |AB||DC| + |AD||BC|$.





I leave this for you to prove.

Hint: If you look at the equations labelled (1) and (2) in Problem 3 you should notice that we haven't used the fact that the triangle ABC is equilateral until the next line. Note that $ABCP$ is a cyclic quadrilateral.

5. The converse of Ptolemy's theorem is also true. Try proving that.

Note: Another theorem I can't resist mentioning is Casey's Theorem which is a generalisation of Ptolemy's theorem. You will find an account of it in Wikipedia. The Irish mathematician John Casey (1820-1891) was a professor of mathematics in the Catholic University of Ireland - UCD can trace its history back to that.

I've added a list of popular maths books that some of you might be interested in reading.

My best wishes to you all. I hope you continue to enjoy maths. Mary Hanley

Popular maths books

Mary Hanley

The books listed here can be enjoyed by people who have an interest in maths without having studied maths at third level.

- *The Simpsons and Their Mathematical Secrets* (2013), by Simon Singh,
- *Alex's Adventures in Numberland* (2010), by Alex Bellos,
- *Fermat's Last Theorem* (1997), by Simon Singh,
- *The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography* (2000), by Simon Singh,
- *The Music of the Primes* (2003), by Marcus du Sautoy,
- *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics* (2003), by John Derbyshire.